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BIFURCATION BEHAVIOUR OF BILAYERED TUBES SUBJECTED TO UNIFORM SHRINKAGE UNDER PLANE STRAIN CONDITION

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Abstract—Nonaxisymmetric bifurcation behaviour of bilayered tubes subjected to uniform shrinkage at the external surface under plane strain conditions has been investigated and compared with that of single tubes. The influence of the thickness ratio and the ratios of material properties upon the bifurcation behaviour has been clarified. The yield stress ratio and hardening exponent ratio substantially affect the bifurcation mode with long wavelength.

Surface-type bifurcation depends entirely on the material characteristics of the inner tube, so that the surface-type bifurcation point of a bilayered tube nearly coincides with that of a single tube with the same material properties as those of the inner tube.

INTRODUCTION

When a thick-walled bilayered tube is deformed axisymmetrically up to a certain limit through a frictionless die as in the tube drawing and sinking processes, a bifurcation from the fundamental axisymmetric deformation to nonaxisymmetric deformation becomes possible. This bifurcation can be represented as the onset of circumferential waves with long wavelength throughout the thickness of the tube or with short wavelength (referred to as surface-type bifurcation) over the traction-free inner surface. For certain combinations of material properties and dimensions of the tubes, the surface type bifurcation—surface instability—is preferred.

As the tube deforms further beyond this bifurcation point, the surface unevenness will grow considerably and the nonuniform deformation will propagate deeply beneath the traction-free inner surface. For a given tube with little ductility, this nonuniform deformation may lead to the development of deformation localization into a narrow shear band, which may govern the final fracture of the tube. An analysis to determine the condition for the onset of such bifurcation and the corresponding mode can be carried out within the framework of Hill's (1958) bifurcation analysis, and the localization of deformation after the onset of such bifurcation can be traced by the post-bifurcation analysis of deformation.

Tomita *et al.* (1985) analysed the influences of material properties and dimensions of the tubes upon the bifurcation behaviour of a single tube subjected to uniform shrinkage at the external surface and Kim *et al.* (1989) clarified the localization of the deformation into a narrow shear band. Recently, due to the rapid increase of composite materials for special purposes, a number of investigations of the deformation characteristics of composite materials relating to real metal forming processes such as upsetting (Dorris and Nemat-Nasser, 1980) and plate rolling (Stief, 1987) has been performed. However, few theoretical investigations have been performed to clarify the deformation behaviour of bilayered tubes during the forming processes.

In this paper our attention is directed toward the bifurcation behaviour of bilayered tubes subjected to a uniform drawing or sinking process through a frictionless die. In order to clarify the influence of material property ratios and the thickness ratio upon bifurcation behaviour, the above-mentioned process is simplified to the process of uniform shrinkage at the external surface under plane strain. With Hill's (1958) general theory of bifurcation and uniqueness in elastic–plastic solids, the bifurcation behaviour of bilayered tubes subjected to uniform shrinkage at the external surface under the plane strain condition are investigated numerically and compared with those of single tubes.

The tube is assumed to be characterized by the J_2 corner theory of Christoffersen and Hutchinson (1979). In the bifurcation analysis, the loading path during axisymmetric deformation is nearly proportional and Budiansky's (1959) total loading condition is satisfied everywhere, so that the material of the tubes is assumed to be characterized by the hypoelastic J_2 deformation theory by Needleman and Tvergaard (1977), a special case of which was given by Storen and Rice (1975). Furthermore, the material of the tubes is conveniently assumed to be incompressible in order to obtain a closed form solution to the fundamental axisymmetric deformation problem.

BASIC ANALYSIS

The governing equations for an elastic–plastic boundary value problem are given within the context of large strain theory. An updated Lagrangian formulation of the field and constitutive equation is employed. Consider a body with volume V and surface S subjected to a velocity constraint on S_v and a nominal traction rate $\dot{P}_t$ on S_t. In the absence of body forces, the solution for an equilibrium state at the current configuration can be determined by the virtual work principle,

$$\int_V (\dot{S}_{ij} + \sigma_{mj} v_{i,m}) \delta v_{i,j} \, dV = \int_{S_t} \dot{P}_i \delta v_i \, dS, \tag{1}$$

where the virtual velocity δv_i satisfies the homogeneous boundary condition over the surface S_v. S_{ij} is Kirchhoff stress, which is identical to Cauchy stress σ_{ij} in the current configuration. $(\dot{\cdot})$ is the material time derivative of ( ), and $(\;)_{,j}$ denotes partial differentiation with respect to the current Cartesian coordinate x_j.

As long as the deformation is sufficiently small, the elastic–plastic boundary value problem has a unique solution, which is referred to a fundamental solution. When the deformation reaches a certain limit, bifurcation from a fundamental solution to a second solution becomes possible. The condition for the onset of bifurcation and the corresponding mode can be found through the use of Hill's (1958) general theory of bifurcation. This theory states that the solution is not unique when a nontrivial solution can be found for the eigenvalue problem given by the following variational equation

$$\delta I = 0, \qquad I = \int_V (\Delta \dot{S}_{ij} + \sigma_{mj} \Delta v_{i,m}) \Delta v_{i,j} \, dV. \tag{2}$$

Here, Δ denotes the difference between the fundamental solution and the second solution. The stress rate $\dot{S}_{ij}$ is related to the strain rate $\dot{\varepsilon}_{kl}$ by the linear comparison solid of Hill (1958) in which elastic unloading from the plastic zone is excluded, and thus the plastic branch of the constitutive tensor is employed for the current plastic zone. Therefore the lower bound of the bifurcation point for the elastic–plastic solid is given by the bifurcation point for the linear comparison solid.

CONSTITUTIVE EQUATIONS

The constitutive equation employed here is that expressed by the J_2 corner theory of Christoffersen and Hutchinson (1979) which permits the development of a corner on the yield surface at the current loading point. The formation of a corner on the yield surface has received considerable application in studies of post-bifurcation phenomena with a strong deviation from proportional loading in the plastic range, as observed by Tvergaard *et al.* (1981), Larrson *et al.* (1982), Tomita *et al.* (1986) and Kim *et al.* (1989). In the J_2

corner theory the instantaneous moduli for nearly proportional loading are chosen equal to those of the large strain generalization of J_2 deformation theory and for increasing deviation from proportional loading the moduli increase monotonically until they coincide with the linear moduli for elastic loading.

The loading path during fundamental axisymmetric deformation is one that is nearly proportional, and satisfies the total loading condition of Budiansky (1959) everywhere, so that the hypoelastic J_2 deformation theory described by Needleman and Tvergard (1977) is employed. The assumption of material incompressibility makes it convenient to obtain a closed form solution to the fundamental axisymmetric deformation problem. The constitutive equation of hypoelastic J_2 deformation theory can then be simplified as follows

$$\begin{aligned} \nabla\sigma'_{ij} &= D_{ijkl}\dot{\varepsilon}_{kl}, \quad \dot{S}_{ij} = (D_{ijkl} - F_{ijkl})\dot{\varepsilon}_{kl}, \\ D_{ijkl} &= (E_s/3)(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) - \eta(E_s - E_t)(\sigma'_{ij}\sigma'_{kl}/\sigma_e^2), \\ F_{ijkl} &= (\sigma_{il}\delta_{kj} + \sigma_{kj}\delta_{il} + \sigma_{jl}\delta_{ik} + \sigma_{ik}\delta_{jl})/2, \\ \dot{\varepsilon}_{kl} &= (v_{k,l} + v_{l,k})/2, \end{aligned} \tag{3}$$

$$\eta = 1: \text{ plastic deformation}, \quad \eta = 0: \text{ elastic deformation},$$

where σ'_{ij} and σ_e are respectively the Cauchy deviatoric stress and the effective von Mises stress, $\sigma_e = \sqrt{(3\sigma'_{ij}\sigma'_{ij}/2)}$. $\nabla\sigma'_{ij}$ is the Jaumann co-rotational rate of the Cauchy deviatoric stress σ'_{ij} and $\dot{\varepsilon}_{kl}$ is the Eulerian strain rate tensor. δ_{ij} is the Kronecker delta. D_{ijkl} are the instantaneous moduli and F_{ijkl} is the fourth order tensor. E_s and E_t are respectively the secant and tangent moduli for the uniaxial true stress–logarithmic strain curve at the current value of σ_e. Furthermore, the constitutive equation (3) can be expressed in terms of the rate of Cauchy deviatoric stress $\nabla\sigma'_i$ and logarithmic strain rate $\dot{\varepsilon}_i$ relative to the principal axes of deformation

$$\nabla\sigma'_i = (2E_s/3)\dot{\varepsilon}_i - \eta(E_s - E_t)(\sigma'_i\sigma'_k/\sigma_e^2)\dot{\varepsilon}_k. \tag{4}$$

In the present case of uniform shrinkage, in which the principal axes of logarithmic strain do not rotate relative to the material and coincide with the cylindrical coordinate axes, the constitutive equation (4) can be integrated to give the deviatoric true stress and logarithmic strain relation in principal axes as in Neale (1981), as follows

$$\sigma'_i = 2E_s\varepsilon_i/3, \quad (i = 1, 3). \tag{5}$$

The principal logarithmic strains ε_i are related to the principal stretches λ_i in the form $\varepsilon_i = \ln\lambda_i$. Due to the material incompressibility the relation of $\lambda_1\lambda_2\lambda_3 = 1$ is preserved. The material of each tube is characterized by the following true stress σ_i—logarithmic strain ε_i curve in uniaxial tension:

$$\varepsilon_i/\varepsilon_{yi} = \begin{cases} \sigma_i/\sigma_{yi}, & \sigma_{yi} \geq \sigma_i, \\ (\sigma_i/\sigma_{yi})^{n_i}, & \sigma_{yi} < \sigma_i, \end{cases} \tag{6}$$

$$i = 1: \text{ inner tube}, \quad i = 2: \text{ outer tube},$$

where σ_{yi} $(= E_i\varepsilon_{yi})$ is the yield stress and ε_{yi} is the yield strain. E_i is Young's modulus and n_i is the work hardening exponent.

PRE-BIFURCATION ANALYSIS

Consider the prebifurcation state of a bilayered tube subjected to uniform shrinkage at the external surface under plane strain conditions. The internal, external and arbitrary radii in the undeformed state are A, B and R respectively, and in the deformed state correspondingly a, b and r. C and c denote the radius of the interface between the inner

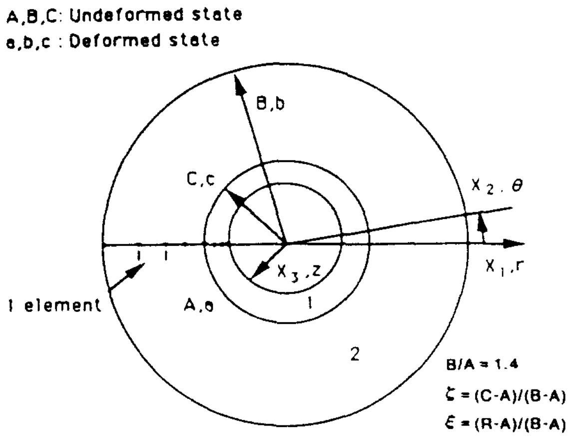

Fig. 1. Coordinate system of the bilayered tube and finite element mesh for bifurcation analysis.

tube 1 and the outer tube 2 in the undeformed and deformed states respectively. The cylindrical coordinate system wherein the indices are identified as $x_1 = r$, $x_2 = \theta$ and $x_3 = z$ is shown in Fig. 1. Since the tube deforms axisymmetrically in the prebifurcation state, the stress state can be expressed as a function of r only and thus the equilibrium equation for axisymmetric deformation becomes

$$\mathrm{d}\sigma_r/\mathrm{d}r + (\sigma_r - \sigma_\theta)/r = 0. \tag{7}$$

Due to the material incompressibility and the plane strain condition the principal stretches at r with original radius R are found to be

$$\lambda_r = 1/\lambda, \quad \lambda_\theta = \lambda, \quad \lambda_z = 1 \tag{8}$$

with $\lambda = r/R$.

Using the constitutive equation (5), the principal strains and the equivalent stress are expressed as

$$\begin{gathered} \varepsilon_r = (3/2)(\sigma_r'/E_s), \quad \varepsilon_\theta = (3/2)(\sigma_\theta'/E_s), \quad \varepsilon_z = 0, \\ \sigma_e = (\sqrt{3}/2)(\sigma_r - \sigma_\theta) = -(2/\sqrt{3})E_s \ln\lambda. \end{gathered} \tag{9}$$

The secant modulus E_{si} $(= \sigma_e/\varepsilon_e)$ and tangent modulus E_{ti} $(= \mathrm{d}\sigma_e/\mathrm{d}\varepsilon_e)$ for the uniaxial true stress–logarithmic strain curve at σ_e are given by

$$\begin{gathered} E_{si} = \begin{cases} E_i, & \sigma_{yi} \geqslant \sigma_e, \\ E_i(\sigma_e/\sigma_{yi})^{1-n_i}, & \sigma_{yi} < \sigma_e, \end{cases} \\ E_{ti} = \begin{cases} E_i, & \sigma_{yi} \geqslant \sigma_e, \\ (E_i/n_i)(\sigma_e/\sigma_{yi})^{1-n_i}, & \sigma_{yi} < \sigma_e, \end{cases} \\ i = 1; \text{ inner tube}, \quad i = 2; \text{ outer tube}. \end{gathered} \tag{10}$$

Under the assumption of fully plastic deformation of the bilayered tube, the equilibrium equation (7) for the axisymmetric deformation with the field equations (8)–(10) gives the fundamental stress distributions as follows:

inner tube 1:

$$\sigma_{r1} = -A_1 \int_a^r (1/r)(\ln(R/r))^{1/n1}\,dr, \quad \sigma_{\theta 1} = \sigma_{r1} - A_1(\ln(R/r))^{1/n1}, \tag{11}$$

outer tube 2:

$$\sigma_{r2} = -A_1 \int_a^c (1/r)(\ln(R/r))^{1/n1}\,dr - A_2 \int_c^r (1/r)(\ln(R/r))^{1/n2}\,dr,$$

$$\sigma_{\theta 2} = \sigma_{r2} - A_2(\ln(R/r))^{1/n2}, \quad \sigma_{zi} = (\sigma_{ri} + \sigma_{\theta i})/2,$$

$$A_i = (4E_i^{1/ni}/3)(2/(\sqrt{3}\sigma_{yi}))^{1/ni-1}, \quad i = 1, 2.$$

The stress distribution during the axisymmetric deformation is obtained by numerical integration of eqn (11) with the condition that the volume remains constant:

$$a = \sqrt{(b^2 - B^2 + A^2)}, \quad r = \sqrt{(b^2 - B^2 + R^2)}. \tag{12}$$

BIFURCATION ANALYSIS

When the tube is shrunk beyond a certain limit under the same boundary conditions, bifurcation from the fundamental axisymmetric deformation to nonaxisymmetric deformation becomes possible. The bifurcation point and corresponding mode can be obtained by using Hill's (1958) general theory of uniqueness and bifurcation in the elastic–plastic solids.

Since the condition of material incompressibility is preserved throughout the whole deformation including that ruling throughout the bifurcation solution, the physical components of admissible velocity in the radial and circumferential direction of the tube are defined by a stream function ϕ such that

$$\Delta v_r = -(1/r)(\partial\phi/\partial\theta), \quad \Delta v_\theta = \partial\phi/\partial r. \tag{13}$$

Consequently, the nonzero components of the velocity gradient and strain rate associated with the admissible velocity are

$$\Delta v_{r,r} = (1/r^2)(\partial\phi/\partial\theta) - (1/r)(\partial^2\phi/\partial r\,\partial\theta), \quad \Delta v_{\theta,\theta} = -\Delta v_{r,r},$$

$$\Delta v_{r,\theta} = -(1/r^2)(\partial^2\phi/\partial\theta^2) - (1/r)(\partial\phi/\partial r), \quad \Delta v_{\theta,r} = \partial^2\phi/\partial r^2,$$

$$\Delta\dot{\varepsilon}_{rr} = \Delta v_{r,r} = (1/r^2)(\partial\phi/\partial\theta) - (1/r)(\partial^2\phi/\partial r\,\partial\theta), \quad \Delta\dot{\varepsilon}_{\theta\theta} = -\Delta\dot{\varepsilon}_{rr},$$

$$\Delta 2\dot{\varepsilon}_{r\theta} = -(1/r^2)(\partial^2\phi/\partial\theta^2) - (1/r)(\partial\phi/\partial r) + \partial^2\phi/\partial r^2. \tag{14}$$

For the problem considered here, the stresses associated with the fundamental solution are in the plastic zone throughout their whole range, and it is assumed that the stress rate at the bifurcation point does not strongly deviate from the whole loading range of the fundamental solution. The moduli D_{ijkl} in eqn (3) which are identified with the help of the fundamental stress distribution can therefore be used in the bifurcation analysis. For the case in which the stress rate deviates from the whole loading range, employment of the same moduli D_{ijkl} in eqn (3) may provide a lower bound to the bifurcation point. In the same manner as in a single tube (Tomita *et al.*, 1985), with constitutive equation (3), and velocity gradient and strain rate in eqn (14), Hill's variational equation (2) may be specified for the present problem:

$$\delta I = 0$$

$$I = \int_0^{2\pi}\int_a^c [B_1\Delta\dot{\varepsilon}_{rr}^2 + C_1(2\Delta\dot{\varepsilon}_{r\theta})^2 + \sigma_{r1}\Delta v_{\theta,r}^2 + \sigma_{\theta 1}\Delta v_{r,\theta}^2]\, r\,\mathrm{d}r\,\mathrm{d}\theta + \int_0^{2\pi}\int_c^b [B_2\Delta\dot{\varepsilon}_{rr}^2 + C_2(2\Delta\dot{\varepsilon}_{r\theta})^2 + \sigma_{r2}\Delta v_{\theta,r}^2 + \sigma_{\theta 2}\Delta v_{r,\theta}^2]\, r\,\mathrm{d}r\,\mathrm{d}\theta, \quad (15)$$

$$B_i = (4/3)E_{ti} - \sigma_{ri} - \sigma_{\theta i}, \quad C_i = E_{ti}/3 - (\sigma_{ri} + \sigma_{\theta i})/2, \quad i = 1, 2.$$

Even when the bilayered tube deforms plastically, the values of $B_1 \sim C_2$ and $\sigma_{r1} \sim \sigma_{\theta 2}$ are dependent on the material properties of the bilayered tube. These values thus affect the bifurcation behaviour of the bilayered tube. The boundary conditions for the admissible velocity components Δv_r, Δv_θ are expressed as

$$\Delta v_r|_{\theta=0} = \Delta v_r|_{\theta=2\pi}, \quad \Delta v_\theta|_{\theta=0,2\pi} = 0. \quad (16)$$

For these boundary conditions the stream function ϕ can then be represented in terms of a Fourier sine series in the θ direction:

$$\phi = \sum_{m=0}^{\infty} \phi_m(r)\sin m\theta, \quad (17)$$

where ϕ_m is the amplitude of the bifurcation mode—a function of r only—and m is the mode number of the bifurcation, the circumferential wave number. It should be noted that the special case of $m = 0$ in eqn (17) leads to the axisymmetric deformation of the fundamental solution. Substituting eqn (17) into bifurcation functional, I, of eqn (15), integrating with respect to θ, and noting the orthogonality of the trigonometric functions, the bifurcation functional, I, is expressed in a completely separated form with respect to the bifurcation mode, as follows:

$$I = \sum_{m=0}^{\infty} I_m, \quad (18)$$

$$I_m = \pi\int_a^c [B_1 m^2\{(1/r^2)\phi_m - (1/r)\phi_m'\}^2 + C_1\{(m^2/r^2)\phi_m - (1/r)\phi_m' + \phi_m''\}^2 + \sigma_{r1}(\phi_m'')^2 + \sigma_{\theta 1}\{(m^2/r^2)\phi_m - (1/r)\phi_m'\}^2]r\,\mathrm{d}r + \pi\int_c^b [B_2 m^2\{(1/r^2)\phi_m - (1/r)\phi_m'\}^2 + C_2\{(m^2/r^2)\phi_m - (1/r)\phi_m' + \phi_m''\}^2 + \sigma_{r2}(\phi_m'')^2 + \sigma_{\theta 2}\{(m^2/r^2)\phi_m - (1/r)\phi_m'\}^2]r\,\mathrm{d}r,$$

with $(\ )' = \mathrm{d}(\ )/\mathrm{d}r$ and $(\ )'' = \mathrm{d}^2(\ )/\mathrm{d}r^2$.

In obtaining the stationary condition for the bifurcation functional I of eqn (18), line finite elements with interpolation functions of Hermitian type which assure the C^1 continuity on the element boundaries are employed to approximate the amplitude of the bifurcation mode within an element. After a lengthy but straightforward calculation, we finally arrive at an approximate bifurcation functional:

$$I_m = \{\Delta\delta_m\}^T[K_m]\{\Delta\delta_m\}, \quad (19)$$

where $\{\Delta\delta_m\}$ denotes the values of ϕ_m and ϕ_m' at the nodal points. The matrix $[K_m]$ can be determined at the same line in the finite element method and depends on the current stressses, bifurcation mode number, m, interpolation functions and their derivatives with respect to radial direction. The stationary condition for the approximate bifurcation functional leads to homogeneous equations with regard to the magnitude of $\{\Delta\delta_m\}$:

$$[K_m]\{\Delta\delta_m\} = 0, \quad m = 1, 2, 3, \ldots \tag{20}$$

If one of these homogeneous equations has a nontrivial solution, bifurcation may take place. The associated bifurcation mode is obtained as the eigenmode of the homogeneous equations. In the numerical calculation of the fundamental solution the external surface of the tube shown in Fig. 1 is continuously drawn uniformly and the corresponding stresses are determined by eqns (11) and (12). After each incremental drawing a check is made for the vanishing of the determinant of $[K_m]$, the matrices of coefficients, for several bifurcation mode numbers m:

$$\det[K_m] = 0, \quad m = 1, 2, 3 \ldots \tag{21}$$

Usually, when the sign of $\det[K_m]$ changes at a specific incremental step, an iterative method is used to determine an accurate bifurcation point associated with the vanishing determinant.

NUMERICAL RESULTS AND DISCUSSION

A schematic view of the bilayered tube in a cylindrical coordinate system is shown in Fig. 1. Thickness ratios, $\zeta = (C-A)/(B-A)$, of magnitudes 0.05 and 0.25 are considered, representing bilayered tubes with thin and thick inner tubes respectively. These two thickness ratios can show the effect of thickness ratio on the bifurcation behaviour of bilayered tubes. In discretizing the tube into line finite elements with two nodes, to clarify the bifurcation behaviour, the size of elements adopted is small near the traction-free inner surface and the interface between the inner tube 1 and outer tube 2, whereas the element size increases exponentially away from the interface in the radial direction. When we take the total number of elements to be 179, the corresponding smallest element has a size of the order of 1/5000 of the thickness of the bilayered tube. This element discretization captures the abrupt change of the short-wavelength mode near the traction-free inner surface and the interface between inner and outer tubes.

Since the traction-free inner surface is highly compressed under uniform shrinkage at the external surface, the deformation of the inner surface is apt to bifurcate into the deformation of the short-wavelength mode. The surface type bifurcation occurs when the tube first meets the following condition due to Hill and Hutchinson (1975) for instability of an incompressible material under a uniform plane strain field

$$\varepsilon_2[1-\sqrt{\{(1-2\varepsilon_2)/(1+2\varepsilon_2)\}}] = 1/n \tag{22}$$

with $\varepsilon_2 = -\ln(A/a)$.

The effect of the hardening exponent on the bifurcation behaviour of a single tube is shown in Fig. 2 to compare it with that of bilayered tubes. The critical bifurcation in the long-wavelength mode is defined as the bifurcation occurring first in the deformation history. The bifurcation strain for the surface type bifurcation corresponding to mode number $m = 500$ coincides with that obtained from eqn (22) up to three significant digits, as was the case in Tomita *et al.* (1985). According to Fig. 2 it is verified that the critical bifurcation mode number for long wavelength is independent of the value of the hardening exponent n and occurs in the range 6–7. As the hardening exponent increases, the fluctuation of bifurcation strain $-\ln(b/B)$ with change of mode number becomes remarkable. It is certain that the surface type bifurcation is the first instability encountered in the deformation history of tubes with low hardening exponent, whereas the long-wavelength mode bifurcation is that for the tubes with high hardening exponent.

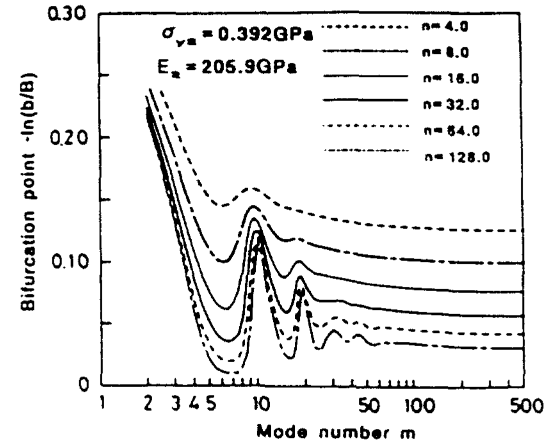

Fig. 2. Mode number m versus bifurcation strain $-\ln(b/B)$ for single tube.

For two thickness ratios, ζ, the effects of material property ratios such as the hardening exponent ratio $\alpha = n_1/n_2$, yield stress ratio $\beta = \sigma_{y1}/\sigma_{y2}$ and Young's modulus ratio $\gamma = E_1/E_2$ on the bifurcation behaviour are investigated. α, β and γ vary relatively to the properties of the external tube which are specified as $E_2 = 21000$ Kgf mm^{-2} (205.9 GPa), $\sigma_{y2} = 40$ Kgf mm^{-2} (0.392 GPa) and $n_2 = 16$.

Figure 3 illustrates the dependence of the bifurcation point on the hardening exponent ratio. As can be seen in the figure, the effect of a bilayered structure is substantial in bifurcation with the long-wavelength mode. For the tubes with $n_1 < n_2$ ($\alpha < 1$) the deformation is destabilized and the onset of bifurcation is accelerated compared with that of a single tube with the same material properties as either the inner or outer tube. The surface type bifurcation point nearly coincides with that of a single tube with the same material properties as the inner tube. This is remarkable in the case of a thick inner tube. It is therefore considered that the surface type bifurcation occurring at the traction-free inner surface is mainly governed by the material characteristics of the inner tube. Furthermore, there is a specific bifurcation mode number with a nearly equal bifurcation point at which an abrupt transition from the long-wavelength mode to the short-wavelength mode occurs. This bifurcation mode number decreases as the thickness of the inner tube increases. This is partly attributable to the decay rate of the bifurcation mode amplitude with respect to the radial direction for different mode numbers.

Figure 4 depicts the effect of β on the bifurcation behaviour. In the case of a single tube, the terms associated with the yield stress σ_y can be taken out of the integrand of eqn

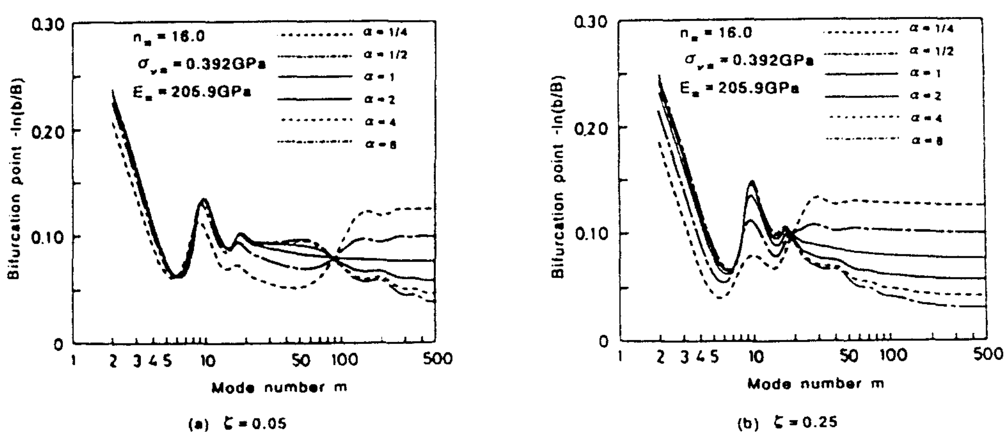

Fig. 3. Effect of hardening ratio α on bifurcation behaviour of bilayered tubes with (a) $\zeta = 0.05$ and (b) $\zeta = 0.25$.

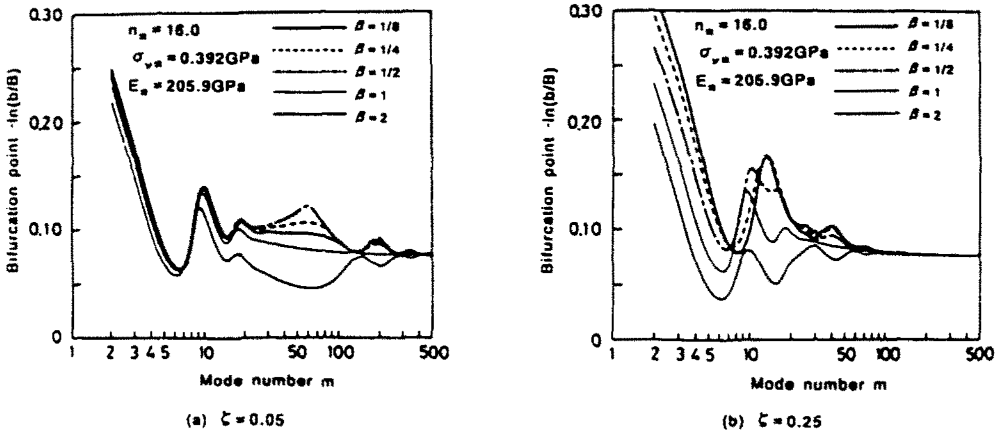

Fig. 4. Effect of yield stress ratio β on bifurcation behaviour of bilayered tubes with (a) $\zeta = 0.05$ and (b) $\zeta = 0.25$.

(18), and bifurcation is therefore independent of σ_y (Tomita *et al.*, 1985). However, as expected from the bifurcation functional, I, eqn (18), the yield stress ratio β affects the bifurcation behavior of bilayered tubes. The influence of β on bifurcation with the long-wavelength mode is essentially notable in tubes with $\sigma_{y1} < \sigma_{y2}$ ($\beta < 1$). For a bilayered tube with a relatively thin inner tube, the difference in material properties affects the bifurcation behavior up to higher modes. This might be attributable to the characteristic features of the decay rate of the bifurcation mode. As far as the present material characteristics are concerned, bifurcation with the long-wavelength mode is critical.

With regard to the effect of Young's modulus ratio, γ, on the bifurcation behaviour, since the whole tube deforms plastically, γ did not notably affect the bifurcation behaviour at any mode number.

Figures 5 and 6 show the amplitude of velocity in the circumferential direction associated with the bifurcation mode, $V_\theta^* = \partial\phi_m/\partial r$, for tubes with $\zeta = 0.05$ and different hardening exponent ratios. Although the bifurcation mode with the long-wavelength strongly depends on the mode number, here, V_θ^* for $m = 5$ is depicted in Fig. 5. The maximum V_θ^* is normalized to 1.0. There is no distinct effect of the bilayered structure in the bifurcation mode shape. The mode with large amplitude has a tendency to localize in the vicinity of the inner surface as the mode number increases. V_θ^* with $m = 500$, surface type bifurcation, for $\alpha = 1$, 0.5, 0.263 and 0.25 is shown in Fig. 6. In the figure, $\zeta = 0.05$ denotes the interface between the inner and outer tubes. For the single tube, the position of the maximum amplitude value of the bifurcation mode occurs at the traction-free surface. The amplitude of the bifurcation mode decays exponentially from that at the position of maximum amplitude. However, for the bilayered tube, the ratio of the hardening exponent, α, affects the shape of the bifurcation mode. As α decreases, the position of maximum amplitude of the bifurcation mode moves gradually away from the traction-free inner surface. When $\alpha = 0.25$ the maximum amplitude occurs within the outer tube, and the effect of nonuniform deformation extends to deep within the outer tube. From this, it can be conjectured that the strain localization starts

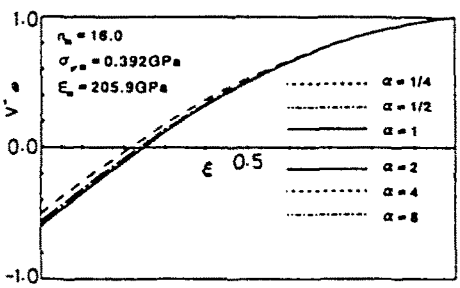

Fig. 5. Normalized eigenmode v_θ^* with long wavelength.

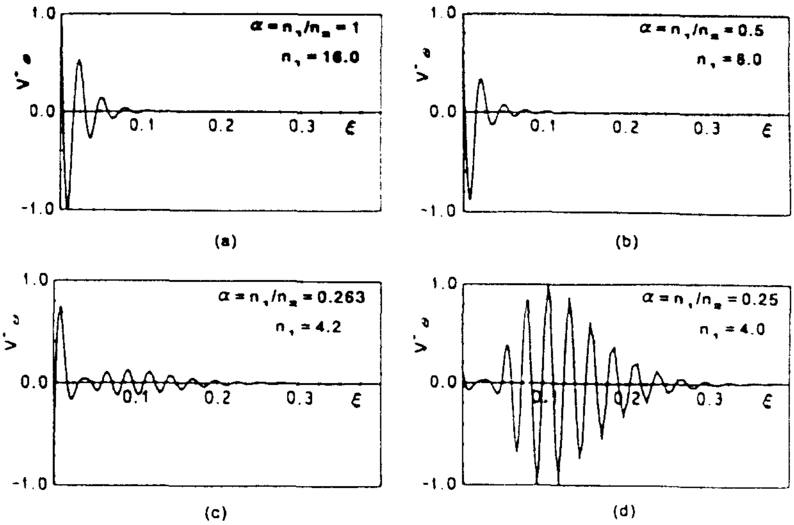

Fig. 6. Normalized eigenmode v_a^* with short wavelength for single layered tube (a), for bilayered tubes with $\zeta = 0.05$ (b) $\alpha = 0.5$ (c) $\alpha = 0.263$ (d) $\alpha = 0.25$.

near the boundary plane and then extends either into or away from the plane. A post bifurcation analysis will need to be performed before this can be discussed in greater detail.

CONCLUSIONS

The bifurcation behaviour of bilayered tubes subjected to uniform shrinkage at the external surface under plane strain condition was analysed numerically, and the influence of thickness ratio and material property ratios such as hardening exponent α, yield stress β and Young's modulus γ upon the bifurcation behaviours was investigated.

According to the result of the present bifurcation analysis, surface type bifurcation with the short wavelength depends entirely on the material characteristics of the inner tube, that the surface type bifurcation point of bilayered tube nearly coincides with that of a single tube having the same material properties as those of the inner tube. However, the ratios of material properties substantially affect the long-wavelength bifurcation mode. The deformation of a bilayered tube composed of an inner tube with low hardening exponent ($\alpha < 1$) or with high yield stress ($\beta > 1$) are destabilized, and the onset of bifurcation is accelerated. It is also concluded that the position of the maximum amplitude of the bifurcation mode moves gradually from the traction-free inner surface into the outer tube as the hardening exponent ratio decreases. This implies the possibility of onset of localization starting from the surface of the boundary between different materials.

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